CHAPTER 2
STATICALLY INDETERMINATE STRUCTURES

2-1- INTRODUCTION

It is customary to divide structures into statically determinate and statically indeterminate. By a statically determinate system we mean a system for which all the reactions of supports can be determined by means of equations of equilibrium, and the internal forces also can be found by method of sections.

![Plane Structure](image1)

![Space Structure](image2)

**Fig. (2-1) Statically Indeterminate Structures**

Consider a structure subjected to several forces in space, this structure will be in equilibrium if the components of the resultant in the three orthogonal directions x, y, and z and equal to zero;

\[
\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0
\]

\[
\sum M_x = 0, \quad \sum M_y = 0, \quad \sum M_z = 0
\]

Thus for space structure subjected to external six equations of static equilibrium can be written. For plane structures (acting in the x-y plane) only three of the six equation of static are meaningful. These equations are;

\[
\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M_z = 0
\]

2-2- DEFINITION

A statically indeterminate system means that the reactions and internal forces cannot be analyzed by the application of the equations of static alone. The indeterminacy of the structure may be either external, internal, or both. The space structure is externally indeterminate if the number of the reaction components is more than six. The corresponding number in a plane structure is three.
Some structures are built with intermediate hinges, each hinge provides an additional equation of static equilibrium and allows the determination of additional reaction component. For instance, the frame given in figure with three intermediate hinges is a statically determinate structure with six reaction components.

Thus, for structures with intermediate hinges, the degree of externally indeterminacy depends on the difference between the number of reactions and the available equations of static equilibrium.

Let us consider structures which are externally statically determinate but internally indeterminate. For instance, closed plane frames given in the figure, each closed part gives three degrees of indeterminacy. The number of indeterminacy decreases by introducing intermediate hinges. Each intermediate hinge removes the bending moment at the respective point.

The space frame given in the figure (2-4) has six unknown reactions at each support (total 24 reactions acting at the four supports). The frame is 18 times externally indeterminate. If the reactions are known, then the internal forces in the four columns can be computed, but the beams forming the closed frame (e,f,g,h) cannot be analyzed.
Cutting one of the beams at any section makes it possible to determine the internal forces in all members of the frame (e,f,g,h). The number of the releases in this case is six, then the structure is internally six times statically indeterminate, and the total degree of indeterminacy is 24.

The truss is called a statically determinate truss, if the reactions and the forces in all the member can be obtained from the conditions of equilibrium alone.

It is called statically indeterminate if these forces can only be determined by taking into account the deformation of the truss. If no equilibrium is possible between external and internal forces the truss is called unstable.

Statically indeterminate trusses are obtained from a statically indeterminate set. Once by the addition of one or more members or components of the reaction without increasing the number of joints, these additional elements are the redundant elements of the truss. Unstable trusses are obtained from statically determinate or indeterminate trusses by taking of one or more necessary members so that at certain joints no equilibrium is possible between external and internal forces as shown in figure (2-5).
2-3- DEGREE OF INDETERMINACY

Consider any statically indeterminate plane truss, the unknown forces are the reaction components \( R \) and the forces in the members \( m \). At each joint two equations of equilibrium can be written:

\[
\Sigma X = 0, \quad \Sigma Y = 0
\]

For a statically determinate system, the number of equations of static is the same as the number of unknowns, i.e.,

\[
2j = m + R
\]

In statically indeterminate trusses, the \((2j)\) equations of equilibrium are not sufficient to find the reactions and the forces in all the members. The degree of indeterminacy is then,

\[
i = (m + R) - 2j
\]

To get these values we must consider the elastic deformation of the truss from which additional equations are derived.

In the case of space trusses, three equations of equilibrium can be written at each joint, i.e.,

\[
\Sigma X = 0, \quad \Sigma Y = 0, \quad \Sigma Z = 0
\]

Thus, the total number of the equations of equilibrium that can be written is \((3j)\). For statically determinate trusses we get,

\[
3j = m + R
\]
Hence, for statically indeterminate trusses the degree of indeterminacy is,

\[ i = (m + R) - 3j \]

At a rigid joint of a plane frame, the unknown forces are the reaction components \( R \) and the internal forces in the members. Each member represents three unknown internal forces. Thus, the total number of unknowns is equal to the sum of internal forces in the members (3m) and the reaction components \( R \).

![Fig. (2-7) The Unknown Forces in Plane Member](image-url)

At each joint three equilibrium equations can be written, two resolution equations and one moment equation. Hence, the internal forces in member can be computed if any three of the six end-forces (\( F_1, F_2, F_3, \ldots, F_6 \)) shown in the figure are known. Hence, the internal forces in the member can be computed. Respectively, the rigid-joined plane frame is statically determinate if,

\[ 3j = 3m + R \]

For statically indeterminate system the degrees of indeterminacy is,

\[ i = (3m + R) - 3j \]

If a rigid joint in the frame is replaced by a hinge, thus the number of the equilibrium equations at this joint will be only two. The bending moment at the end of the members meeting at the joint will vanish. Hence, the total number of unknowns is reduced by one.

We should note that at a rigid joint frames where more than two members meet and one of the members is connected to the joint by a hinge, the number of unknowns is reduced by one, without a reduction in the number of equilibrium equations.

![Fig. (2-8)](image-url)
At a rigid joint of a space frame, for each member three resolution and three moment equations can be written. Hence the internal forces in the member can be computed if any six of the twelve end-forces shown in figure are known. So that each member represents six unknown forces. A space frame is statically determinate if;

\[ 6j = 6m + R \]

and the degree of indeterminacy is;

\[ i = (6m + R) - 6j \]

**Examples**

Determine the number of indeterminacy for the shown three frames shown in fig. (2-9)

![Fig. (2-9)](image)

(a) For the plane frame shown in figure (a);
\[
i = (3m + R) - 3j
\]
\[
= (3 * 3 + 6) - 3 * 4 = 3
\]
Then it is a three times statically indeterminate frame.

b) For the plane frame shown in figure (b);
\[
i = (3m + R) - 3j
\]
\[
= (3 * 7 + 4) - 3 * 6 = 7
\]
Then it is a seven times statically indeterminate frame.

c) For the plane frame shown in figure (b);
\[
i = (6m + R) - 6j
\]
\[
= (6 * 8 + 24) - 6 * 8 = 24
\]
Then it is 24 times statically indeterminate frame.
2-4- KINEMATIC INDETERMINACY

At the supports one of more of the displacement components are known. For instance, the frame given in the figure (2-10) joints c and d have a horizontal displacement \( D_1 \) point c has horizontal displacement \( D_1 \) and rotation \( D_2 \). While point d has horizontal displacement \( D_1 \) and rotation \( D_3 \).

![Fig. (2-10)](image)

Notice that the horizontal displacement at point c and d are equal. The fixation at supports a and b prevent any translation or rotation.

![Fig. (2-11)](image)

The frame given in the figure (2-11) has three degrees of kinematic indeterminacy. The rotations \( D_1, D_2 \) and \( D_3 \) at joints a, b and c.

The frame a b c given in fig. (2-12) has a fixed support at point a and a roller support at point b. The roller support allows point b to displace horizontally \( (D_1) \). Consider that the length of the members ac and cb is constant, then joint c is free to rotate \( D_2 \) and to translate vertically and perpendicularly to member ac the values \( D_3 \) and \( D_4 \).

![Fig. (2-12)](image)

These values \( D_3 \) and \( D_4 \) is a function of \( D_1 \). Hence, the structure has two kinematics indeterminacy \( D_1 \) and \( D_2 \).
2-5- ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES

The object of the analysis is to determine the external and internal forces of the structure. These forces must satisfy the conditions of equilibrium and produce deformations compatible with the continuity of the structure and the support conditions.

Two general methods of approach can be used. The first is the force or flexibility method and the second is the displacement or stiffness method.

The force method treats the member forces as the basic unknowns, whereas the displacement method regards the nodal displacements as the basic unknowns.

If the number of indeterminacy is small, say two or three times, the solution can be done by hand. For larger numbers it is essential to develop and use matrix notation in order to keep control of the very large amount of data.

2-6- FORCE-DISPLACEMENT RELATIONS

Consider the spring shown in the figure (2-13), loaded by a force $P$ and has a displacement $D$.

- **Flexibility**

  Flexibility = displacement per unit force = $\delta$. Fig. (2-13)

  Hence the displacement of the spring under a force $P$ is

  \[ D = \delta * P \]

- **Stiffness**

  Stiffness = force per unit displacement = $K$.

  Hence the force required to produce displacement $D$ is

  \[ P = K * D \]

2-7- DESCRIPTION OF METHODS

To solve statically indeterminate structures by the flexibility or stiffness methods, the following procedures are considered.
Flexibility Method

Release the indeterminate constraints, and the resulting deformation discontinuity calculated. Then redundant actions are then replaced to restore the continuity and the resulting compatibility equations solved for the redundant force actions.

- Stiffness Method

Additional restraints are added to fix all the degrees of freedom and the values of these restraints calculated. The restraints are then removed to allow deformations and restore equilibrium. The resulting equilibrium equations are solved for the displacements and subsequently the force actions are determined.

Example

For example consider the shown one statically indeterminate beam shown in fig.(2-14). This beam has a roller support at point 1 and a fixed support at point 2. The span of the beam \( L = 6.00 \text{ m} \) and it carries a uniformly distributed load of \( 2 \text{ t/m} \).

![Fig. (2-14)](image)

2-7-1 Solution by Flexibility Method

1) Release the reaction at support 1 to reduce to a statically determinate system. The deflection at point 1 will be

\[
D_1 = -\frac{WL^4}{8EI} = -\frac{324}{EI}
\]

2) Apply unit forces \( F_1 = 1.0 \) at point 1 upwards. The deflection at point 1 will be

\[
\delta_{11} = +\frac{PL^3}{3EI} = +\frac{72}{EI}
\]

3) The compatibility condition is that the final deflection at point 1 is zero.

\[
D_1 + F_1 \delta_{11} = 0
\]
From which we get the unknown reaction $F_1$ at joint 1

$$F_1 = 4.5t$$

Knowing the reaction at joint 1, the problem is solved. The final external and internal forces can be determined from the relation;

<table>
<thead>
<tr>
<th>Reaction</th>
<th>$R = R_o + F_1 R_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear</td>
<td>$Q = Q_o + F_1 Q_1$</td>
</tr>
<tr>
<td>Moment</td>
<td>$M = M_o + F_1 M_1$</td>
</tr>
</tbody>
</table>

Flexibility Method

Stiffness Method

2-7-2 **Solution by Stiffness Method**

Analyzing the same beam given in figure 4, the procedure is almost the reverse. This beam has one degree of kinematic indeterminacy $D_1$, the rotation at support 1.

1) Restraining forces equal in number to the degree of kinematic indeterminacy are introduced at the coordinates to prevent the rotation of the joints. In our example the structure is once kinematically indeterminate, it is $D_1$, the rotation at joint 1

To restrain joint 1 against rotation, a moment equal to the fixed end moment will acts at joint 1. Hence we get;

$$F_i = \frac{-WL^2}{12} = -6t.m$$

2) Give unit rotation at joint 1 (notice that the force required to give a unit rotation is known by stiffness $K$).
Due to unit rotation of joint 1 we get:

\[ K_{11} = \frac{4EI}{L} = \frac{4EI}{6} = \text{force at 1 due to unit rotation at 1} \]
\[ K_{21} = \frac{2EI}{L} = \frac{2EI}{6} = \text{force at 2 due to unit rotation at 1} \]

3) Finally for the equilibrium condition the constraint at joint 1 must be removed, i.e., the structure allowed to deformed. If the final rotation at joint 1 is \( D_1 \), the associated moment at 1 will be:

\[ F_1 + K_{11} D_1 = 0 \]
\[ -6 + \frac{4EI}{6} D_1 = 0 \]

Then:

\[ D_1 = \frac{9}{EI} \]

4) Knowing the value of \( D_1 \), the final moment at joint 1 and 2 can be determined.

Joint 1 \( M_1 = 0 \)
Joint 2 \( M_2 = F_2 + K_{21} D_1 \]
\[ = 6 + \frac{2EI}{6} \times \frac{9}{EI} = 9 \text{ t.m} \]

Knowing the moment at 1 and 2, the external forces and internal forces of the beam can be determined.

**2-7-3 Comparison Between the Flexibility and the Stiffness Methods**

![Fig. (2-15)](image-url)
The truss shown in figure (2-15-a) is four times statically indeterminate. This truss has two degrees of kinematic displacement $D_1$ and $D_1$ at joint O. Solving by the force method we get four unknowns (four equations), while solving by the stiffness method we get two unknowns in two equations. Hence, this truss is easier to be solved by stiffness method.

Conversely, the truss shown in figure (2-15-b), is once statically indeterminate and has a single redundant, while it has twelve possible joint displacements (at each joint two degrees of kinematic displacement). Hence it is easy to be solved by the flexibility method.